Inequality 14

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Q> Let a, b, c, d, e be non-negative real numbers such that $a+b+c+d+e = 5$. Prove that,

$$abc + bcd + cde + dea + eab \leq 5$$

Ans:- WLOG $a \leq b \leq c \leq d \leq e$

$$abc + bcd + cde + dea + eab$$

$$= a(e+c)(b+d) - acd + bcd + cde$$

$$= a(e+c)(b+d) + cd(b+e-a)$$

$$\qquad \text{|| } AM^2 \geqslant GM^2 \qquad\qquad \text{|| } AM^3 \geqslant GM^3$$

$$\leq a\left(\frac{b+c+d+e}{2}\right)^2 + \left(\frac{b+e+c+d-a}{3}\right)^3$$

$$= a\left(\frac{5-a}{2}\right)^2 + \left(\frac{5-2a}{3}\right)^3$$

as $a, b, c, d, e \in \mathbb{R}^+$ we get $a \in (0, 5)$

$$a\left(\frac{5-a}{2}\right)^2 + \left(\frac{5-2a}{3}\right)^3 \leq 5$$

$$\Longleftrightarrow \quad \frac{a(25-10a+a^2)}{4} + \frac{125+60a^2-250a-8a^3}{27} \leq 5$$

$$\Longleftrightarrow \quad \frac{27(25a-10a^2+a^3)+4(125+60a^2-250a-8a^3) - 27\times4\times5}{27\times4} \leq 0$$

$$\Longleftrightarrow \quad -5a^3 - 30a^2 - 325a - 40 \leq 0$$

$$\Longleftrightarrow \quad a^3 + 6a^2 + 65a + 8 \geqslant 0$$

as $a \in (0,5)$ this is true

Q> $x, y, z \geqslant 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that,

$$\sqrt{x+y+z} \geqslant \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

Ans:- $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$

$$\Rightarrow \quad -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -2$$

$$\Rightarrow \quad 1 - \frac{1}{x} + 1 - \frac{1}{y} + 1 - \frac{1}{z} = 1$$

$$\Rightarrow \quad \frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 1$$

$$(x+y+z) = (x+y+z)\overbrace{\left(\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z}\right)}^{=1}$$

$$\geq \left(\sqrt{\frac{x-1}{x}}\sqrt{x} + \sqrt{\frac{y-1}{y}}\sqrt{y} + \sqrt{\frac{z-1}{z}}\sqrt{z}\right)^2$$

$$= \left(\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}\right)^2$$

$$\Rightarrow \quad \sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

Q) $a, b, c \in \mathbb{R}^+$ such that $a+b+c = 1$. Prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq 5$$

Ans:—

$$4a+1 < 4a^2+4a+1 = (2a+1)^2. \quad (\text{as } a > 0)$$

$$\Rightarrow \quad \sqrt{4a+1} < 2a+1$$

$$\Rightarrow \quad \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 2(a+b+c) + 3 = 5$$

$x_1 = \sqrt{4a+1} \qquad y_1 = 1$

$x_2 = \sqrt{4b+1} \qquad y_2 = 1$

$x_3 = \sqrt{4c+1} \qquad y_3 = 1$

$$\Rightarrow \quad (x_1^2+x_2^2+x_3^2)(1^2+1^2+1^2) \geq (x_1+x_2+x_3)^2$$

$$\Rightarrow \quad \left(\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1}\right)^2 \leq (4a+1+4b+1+4c+1)(3)$$

$$\Rightarrow \quad \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq \sqrt{3(4+3)} = \sqrt{21} < 5$$

Q> $a, b, c \in \mathbb{R}^+$ with $a+b+c=1$. Prove that,

$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \leq \frac{2a}{b} + \frac{2b}{c} + \frac{2c}{a}$$

Ans:- $$\frac{1+a}{b+c} + \frac{1+b}{c+a} + \frac{1+c}{a+b} = 3 + \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}$$

$$= 3+2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)$$

$$3+2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \leq 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$$

$$\Leftrightarrow \left(\frac{a}{b} - \frac{a}{b+c} + \frac{b}{c} - \frac{b}{c+a} + \frac{c}{a} - \frac{c}{a+b}\right) \geq \frac{3}{2}$$

$$\Leftrightarrow \underbrace{\left(\frac{ac}{b(b+c)} + \frac{ab}{c(c+a)} + \frac{bc}{a(a+b)}\right)}_{S} \geq \frac{3}{2}$$

$$S = \frac{a^2c^2}{abc(b+c)} + \frac{a^2b^2}{abc(c+a)} + \frac{b^2c^2}{abc(a+b)}$$

$$\geq \frac{(ac+ab+bc)^2}{abc(a+b+b+c+c+a)} = \frac{(ab+bc+ca)^2}{2abc(a+b+c)} = \frac{(ab+bc+ca)^2}{2abc}$$

$$\Rightarrow S \geq \frac{a^2b^2+b^2c^2+c^2a^2+2abc}{2abc} = \frac{a^2b^2+b^2c^2+c^2a^2}{2abc} + 1$$

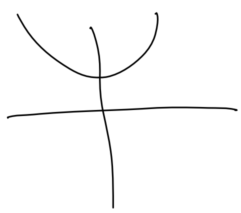
$$s_1 = (ab+bc+ca)^2 - 3abc(a+b+c)$$

$$= a^2b^2+b^2c^2+c^2a^2 - abc(a+b+c)$$

$$= (b^2+c^2-bc)a^2 - bc(b+c)a + b^2c^2$$

$b^2+c^2-2bc \geq 0$

$b^2+c^2-bc \geq bc > 0$



$$\text{Discriminant} = b^2c^2(b+c)^2 - 4(b^2+c^2-bc)b^2c^2$$

$$= b^2c^2(b^2+2bc-c^2)$$

Discriminant $= b^2c^2(b+c) - 4(b^2+c^2-bc)bc$

$$= -3b^2c^2(b^2+2bc-c^2)$$

$$= -3b^2c^2(b-c)^2 \leq 0$$

$\Rightarrow S_1 \geqslant 0$

$\Rightarrow (ab+bc+ca)^2 \geqslant 3abc(a+b+c) = 3abc$

$\Rightarrow \dfrac{(ab+bc+ca)^2}{2abc} \geqslant \dfrac{3}{2}$

$\Rightarrow S \geqslant \dfrac{3}{2}$ $\Rightarrow$ Hence done